Math 579 Fall 2013 Exam 5 Solutions

1. How many surjective functions are there from [8] to [3]?

   There are $3^8 = 6561$ functions, but not all are surjective. 3 map onto a single element. 2^8 map onto only $\{1, 2\}$; however two of these map onto a single element, so there are $2^8 - 2 = 254$ surjective functions to $\{1, 2\}$. Similarly, there are 254 surjective functions to $\{1, 3\}$ and 254 to $\{2, 3\}$. Our answer is $6561 - 3 - 254 - 254 - 254 = 5796$.

2. Let $F(n)$ denote the number of partitions of $[n]$ with no singleton blocks. Recall that $B(n) = \sum_{i=0}^{n} \left\{ {n \atop i} \right\}$. Prove that $B(n) = F(n) + F(n+1)$.

   Combinatorial proof: Consider all partitions of $[n]$ into any number of blocks; this is counted by $B(n)$. Some of these have no singleton blocks; these are counted by $F(n)$. The remainder have at least one singleton block. Put all these (at least 1) singletons together, and add the element $n + 1$. The result is a partition of $n + 1$ that has no singleton blocks. Since this process is reversible (remove $n + 1$, and separate the rest of that part into singletons), these partitions are counted by $F(n + 1)$.

3. For each $n \in \mathbb{N}$, calculate $\sum_{k=1}^{n} k^5$.

   The hardest part is finding the Stirling numbers we need to rewrite k^5 in terms of falling powers via Cor 5.10. We can do this with the addition formula (Thm 5.8) recursively; the result is $k^5 = k^{\underline{1}} + 15k^{\underline{2}} + 25k^{\underline{3}} + 10k^{\underline{10}} + k^{\underline{5}}$. Hence $\sum_{0}^{k+1} k^5 \delta k = \frac{1}{2}(n+1)^{\underline{2}} + 5(n+1)^{\underline{3}} + \frac{25}{4}(n+1)^{\underline{4}} + 2(n+1)^{\underline{5}} + \frac{1}{6}(n+1)^{\underline{6}} - 0$.

4. Find the general solution $y(x)$ to the second-order difference equation $\Delta^2 y = 3^x + x$.

   We first find the general antidifference $\Delta y = \sum 3^x + x^{\underline{1}} \delta x = \frac{1}{2} 3^x + \frac{1}{2} x^{\underline{2}} + C$. We now find the antidifference again to get $y = \frac{1}{4} 3^x + \frac{1}{6} x^{\underline{3}} + C x^{\underline{1}} + D$, for arbitrary $C, D \in \mathbb{R}$.

5. Calculate $\sum_{k=0}^{20} (k^2 - k) 3^{-k}$.

   We will use summation by parts twice; for convenience we drop the endpoints and work with indefinite sums until the end. $u(k) = k^{\underline{2}}, \Delta u(k) = 2k^{\underline{1}}$, $\Delta v(k) = 3^{-k}$, $v(k) = -\frac{3}{2} 3^{-k}$, $Ev(k) = -\frac{3}{2} 3^{-(k+1)} = -\frac{1}{2} 3^{-k}$. Hence $\sum u(k) \Delta v(k) = -\frac{3}{2} k^{\underline{2}} 3^{-k} + \sum k^{\underline{1}} 3^{-k}$. Now to find $\sum k^{\underline{1}} 3^{-k}$ we use $u(k) = k^{\underline{1}}, \Delta u(k) = 1, \Delta v(k) = 3^{-k}, v(k) = -\frac{3}{2} 3^{-k}, Ev(k) = -\frac{1}{2} 3^{-k}$. Hence $\sum k^{\underline{1}} 3^{-k} = -\frac{3}{2} k^{\underline{1}} 3^{-k} + \frac{1}{2} \sum 3^{-k} = -\frac{3}{2} k^{\underline{1}} 3^{-k} - \frac{3}{4} 3^{-k}$. Putting it all together, our answer is $-\frac{3}{2} k^{\underline{2}} 3^{-k} - \frac{3}{2} k^{\underline{1}} 3^{-k} - \frac{3}{4} 3^{-k} = -\frac{3}{4} 3^{-k} (2k^{\underline{2}} + 2k^{\underline{1}} + 1) = -\frac{3}{4} 3^{-k} (2k^2 + 1)$. We evaluate at 21 and 0, then subtract to get $-\frac{3}{4} 3^{-21} (2(21)^2 + 1) + \frac{3}{4} 3^0 (2(0)^2 + 1) = \frac{3}{4} - \frac{2649}{4} 3^{-21}$ or if you're really hardcore $\frac{2615088080}{3485784401}$.